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Effective detection of nonsplit module extensions

Edward S. Letzter[1]

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

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Abstract

Let n be a positive integer, and let R be a finitely presented (but not necessarily finite dimensional) associative algebra over a computable field. We examine algorithmic tests for deciding (1) if every at most-n-dimensional representation of R is semisimple, and (2) if there exist nonsplit extensions of non-isomorphic irreducible R-modules whose dimensions sum to no greater than n.

MSC: 16Z05; 13P99

1. Introduction

If $R = k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$ is a finitely presented algebra over a field k, then it is easy to see that the n-dimensional representations of R amount to solutions to a system of tn^2 commutative polynomial equations in sn^2 variables. Moreover, the n-dimensional irreducible representations of R can also be explicitly parametrized by finite systems of commutative polynomial equations (cf. [1,15]). Consequently, the techniques of computational algebraic geometry (and in particular, Groebner basis methods) can be used to study the n-dimensional representation theory of R (cf. [12,13]); for example, the question of whether or not R has an irreducible n-dimensional representation can be algorithmically decided (when k is computable). In this paper we consider algorithmic approaches to another fundamental question in the representation theory of R: Do there exist nonsplit extensions of finite dimensional R-modules?

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E-mail address: letzter@math.temple.edu (E.S. Letzter).

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We present effective procedures for deciding (1) if every at most n-dimensional representation of R is semisimple (i.e., if there exist no nonsplit extensions of modules whose dimensions sum to no greater than n), and (2) if there exists a nonsplit extension of an m-dimensional irreducible representation of R by a non-isomorphic ℓ-dimensional irreducible representation, for some $\ell + m \leqslant n$. These procedures are indirect—they do not give the exact dimensions of the detected nonsplit extensions. However, precise (and more costly) algorithms can be subsequently derived.

Our basic strategy is to reduce each of the considered representation theoretic decision problems to the problem of deciding whether a particular finite set of commutative polynomials has a common zero. Standard methods of computational algebraic geometry can then be applied (in principle). A brief discussion of the complexity of this approach is given in (2.6). The case when $n = 2$, discussed in Section 5, provides an elementary illustration.

When R is known beforehand to be finite dimensional over k, effective methods for determining a linear basis for the Jacobson radical of R have been given in [8,9,16].

2. Preliminaries

In this section we develop our notation (which will remain fixed for the remainder) and quickly review some necessary background.

2.1. We assume throughout this paper that ℓ, m, and n are positive integers, that k is a field, that K is a field extension of k, that $f_1, \ldots, f_t$ are noncommutative polynomials in the free associative k-algebra $k\{X_1, \ldots, X_s\}$, and that R is the quotient algebra

$$k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle.$$

Let d denote the maximum of the total degrees of the $f_1, \ldots, f_t$.

2.2. (i) We will use the term *indeterminate* only in reference to a variable in an (often tacitly given) commutative polynomial ring. Unless otherwise designated, *polynomial* will refer only to a commutative polynomial.

(ii) Let A be a k-algebra (algebras, modules, and homomorphisms will always be assumed to be unital). If $a_1, \ldots, a_q \in A$, we use $k\{a_1, \ldots, a_q\}$ to denote the k-subalgebra generated by $a_1, \ldots, a_q$.

Recall that every K-algebra automorphism τ of $M_n(K)$ is *inner* (i.e., there exists an invertible matrix Q in $M_n(K)$ such that $\tau(a) = QaQ^{-1}$ for all $a \in M_n(K)$).

(iii) We let $M_n(K)$ denote the ring of $n \times n$ matrices with entries in K, and we let $M_{\ell\times m}(K)$ denote the $M_\ell(K)$–$M_m(K)$-bimodule of $\ell \times m$ matrices. We identify K^n with the left $M_n(K)$-module of $n \times 1$ matrices with entries in K.

Let I_n denote the $n \times n$ identity matrix. When $\ell < n$, we identify I_ℓ with the $n \times n$ matrix $\begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}$. Let $SupDiag_n$ denote the $n \times n$ matrix with 1s on the super-diagonal and 0s elsewhere, and let $SubDiag_n$ denote the transpose of $SupDiag_n$. It is easy to verify that $SupDiag_n$ and $SubDiag_n$ generate $M_n(K)$ as a K-algebra.

(iv) We will use the expression (*n-dimensional*) representation of A only to refer to k-algebra homomorphisms $\rho\colon A \to M_n(K)$; the representation is *irreducible* when $K\rho(A) = M_n(K)$.

This approach allows us to consider the K-representation theory of A while restricting our calculations to k; in our algorithmic procedures below we will assume that k is computable and that K is the algebraic closure of k. (Recall, if K is the algebraic closure of k, that a representation $\rho\colon R \to M_n(K)$ is irreducible—in the preceding sense—if and only if the only $K\rho(R)$-invariant subspaces of K^n are 0 and K^n itself.)

(v) Two representations $\rho, \rho'\colon A \to M_n(K)$ are equivalent (or isomorphic) provided there exists an invertible matrix $Q \in M_n(K)$ such that $\rho'(a) = Q^{-1}\rho(a)Q$ for all $a \in A$. We will say that a representation ρ of A is *semisimple* if $K\rho(A)$ is semisimple as a K-algebra.

2.3. (i) For $1 \leqslant \mu \leqslant s$, let $\mathbf{x}_\mu$ denote the generic $n \times n$ matrix $(x_{ij}(\mu))$ (i.e., the $n \times n$ matrix whose ijth entry is the indeterminate $x_{ij}(\mu)$), and set $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_s)$. Note that R has an n-dimensional representation if and only if the entries of $f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})$ have a common zero.

(ii) (Assume that k is computable and that K is the algebraic closure of k.) Using standard techniques of computational commutative algebra, we can check if $f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})$ have a common zero, and thereby decide whether or not R has an n-dimensional representation. Also, we can always slightly simplify the computations by replacing one of the generic matrices $(x_{ij}(\mu))$ with an upper triangular matrix (i.e., by setting $x_{ij}(\mu) = 0$ for $i > j$). Therefore, this procedure involves tn^2 polynomials, of degree at most d, in $sn^2 - (n^2 - n)/2$ variables. (Of course, the specific relations defining R may allow for further reductions.)

In all of the tests discussed below, we will assume that one of the generic matrices has been similarly replaced with a generic upper triangular matrix.

2.4. (i) Let $\mathscr{P}(n)$ denote the minimum positive integer with the following property: For all positive integers q, and for all $a_1, \ldots, a_q \in M_n(K)$, the K-algebra $K\{a_1, \ldots, a_q\}$ is K-linearly spanned by products of the $a_1, \ldots, a_q$ having length no greater than $\mathscr{P}(n)$. (The identity matrix is a product of length zero.)

It is easy to check that $\mathscr{P}(n) \leqslant n^2 - 1$, and in [14] it is proved that $\mathscr{P}(n)$ is bounded above by a function in $\mathscr{O}(n^{3/2})$.

(ii) Let $\rho : R \to M_n(K)$ be a representation, and set $\Lambda = K\rho(R)$. It follows from (i) that Λ is K-linearly spanned by the images of the monomials (in the X_i) having length no greater than $\mathscr{P}(n)$. Also, the Cayley–Hamilton Theorem tells us that the nth power of a matrix in $M_n(K)$ is a K-linear combination of its lower powers. Therefore, Λ is K-linearly spanned by the image under ρ of

$$\{Y_1^{i_1} \ldots Y_p^{i_p} : Y_1, \ldots, Y_p \in \{1, X_1, \ldots, X_s\};\ i_1 + \cdots + i_p \leqslant \mathscr{P}(n);\ i_1, \ldots, i_p < n\}.$$

2.5. For later comparison, we briefly mention two algorithmic tests for detecting irreducible n-dimensional representation. Let

$$\mathbf{W} = \Big\{\mathbf{w}_1^{i_1} \ldots \mathbf{w}_p^{i_p} : \mathbf{w}_1, \ldots, \mathbf{w}_p \in \{I_n, \mathbf{x}_1, \ldots, \mathbf{x}_s\};\ i_1 + \cdots + i_p \leqslant \mathscr{P}(n);\ i_1, \ldots, i_p < n\Big\}.$$

Assume (for the rest of this subsection) that k is computable and that K is the algebraic closure of k.

(i) (Naive Irreducibility Test) For each choice of $\mathbf{w}_1, \ldots, \mathbf{w}_{n^2} \in \mathbf{W}$ we can construct a subtest that returns "true" if the entries of

$$f_1(\mathbf{x}), \ldots, f_t(\mathbf{x}), \quad y_1\mathbf{w}_1 + \cdots + y_{n^2}\mathbf{w}_{n^2} - SupDiag_n,$$
$$z_1\mathbf{w}_1 + \cdots + z_{n^2}\mathbf{w}_{n^2} - SubDiag_n$$

have a common zero, for indeterminants y_i and z_i. The subtest returns "false" if no common zero exists.

It follows immediately that the following are equivalent: (1) at least one of the possible choices of $\mathbf{w}_1, \ldots, \mathbf{w}_{n^2}$ produces a "true" in the subtest, (2) there exists an irreducible representation $R \to M_n(K)$. (Of course, $SupDiag_n$ and $SubDiag_n$ can be replaced with any pair of matrices in $M_n(k)$ that generate $M_n(K)$ as a K-algebra.)

Note that each subtest involves $(t+2)n^2$ polynomials in $(s+2)n^2-(n^2-n)/2$ variables. The degrees of $2n^2$ of these polynomials will be bounded by $\mathscr{P}(n)+1$, and the remaining degrees will be bounded by d.

(ii) Recall the vth *standard identity*,

$$s_v = \sum_{\sigma \in S_v} (\operatorname{sgn} \sigma) Y_{\sigma(1)} \cdots Y_{\sigma(v)} \in \mathbf{Z}\{Y_1, \ldots, Y_v\}.$$

(See, e.g., [17].) Observe that s_v is multilinear and alternating. Choose $\mathbf{w}_0, \ldots, \mathbf{w}_{2(n-1)} \in W$, and let w be an indeterminate. Consider a test that returns "true" if

$$w\, trace(\mathbf{w}_0 s_{2(n-1)}(\mathbf{w}_1, \ldots, \mathbf{w}_{2(n-1)})) - 1$$

and the entries of $f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})$ have a common zero (and returns "false" otherwise). In [12] it is shown that R has an irreducible n-dimensional representation if and only if at least one of these tests returns a "true".

Each subtest in this procedure will involve tn^2+1 polynomials in $sn^2-(n^2-n)/2+1$ variables. One of these polynomials will have degree $\mathscr{P}(n)^{2n-1}+1$, and the remaining degrees will be bounded by d.

2.6. Kollár's Sharp Effective Nullstellensatz [11] offers a rough method, as follows, to compare the complexities of the algorithms we encounter (cf. [4]).

(i) Let $q, r, d_1 \leqslant \cdots \leqslant d_q$ be positive integers, with no $d_i = 2$, and with $r > 1$. Set

$$\mathscr{D} = \begin{cases} d_1 \ldots d_q, & q \leqslant r, \\ d_1 \ldots d_{r-1} d_q, & 1 < r < q. \end{cases}$$

(ii) Let $g_1, \ldots, g_q \in k[x_1, \ldots, x_r]$, and suppose that $d_i = \deg(g_i)$ for $1 \leqslant i \leqslant q$. In [11] it is shown that $g_1, \ldots, g_q$ have no common zero (over the algebraic closure of k) if and only if there exist $h_1, \ldots, h_q \in k[x_1, \ldots, x_r]$ such that $h_1 g_1 + \cdots + h_q g_q = 1$ and such that the degrees of the $g_i h_i$ are no greater than $\mathscr{D}$. It is further shown in [11], for arbitrarily chosen $g_1, \ldots, g_q$ satisfying the given criteria, that this degree bound is as small as possible.

(iii) Following [4, Section 3] (cf. [5, 1.2.5]), we use $\mathscr{D}$ as a relative measure of the complexity of determining whether $g_1, \ldots, g_q$ have a common zero. (In measuring $\mathscr{D}$ for the systems below, we will simply—and simplistically—assume that the degree of a quadratic polynomial is replaced by a 3 in the appropriate calculation).

(iv) Let u denote the minimum of s and t. For the test deciding whether R has an n-dimensional representation (2.3ii), $\mathscr{D} \leqslant d^{un^2}$.

(v) For convenience, in comparing costs of algorithms we will assume that $\mathscr{P}(n) \geqslant d$.

(vi) For the first irreducibility test (2.5i), we see that $\mathscr{D} \leqslant d^{un^2}(\mathscr{P}(n)+1)^{2n^2}$. For the second (2.5ii), we see that $\mathscr{D} \leqslant d^{un^2}(\mathscr{P}(n)^{2n-1}+1)$.

(vii) Unfortunately, the degree bounds in (iv) and (vi) involve factors no smaller than n raised to a polynomial in n. The degree bounds we will encounter in later sections behave similarly. However, the calculation of $\mathscr{D}$, following [11], does not take into account the specific representation-theoretic sources of the polynomials occurring. We therefore ask: What are the minimum degree complexities of n-dimensional representation-theoretic decision problems?

3. Semisimplicity test

Let A denote a k-algebra.

3.1. Set

$$E_{\ell,m}(K) = \begin{bmatrix} M_\ell(K) & M_{\ell\times m}(K) \\ 0 & M_m(K) \end{bmatrix},$$

a K-subalgebra of $M_{\ell+m}(K)$.

The next result will form the foundation for our semisimplicity test. The proof will follow immediately from (3.7).

3.2 Proposition. *Every at-most n-dimensional representation of A is semisimple if and only if $SupDiag_{\ell+m} \notin K\rho(A)$ for all representations $\rho: A \to E_{\ell,m}(K) \subset M_{\ell+m}(K)$ such that $\ell+m \leqslant n$.*

3.3. We will need some more notation.

(i) Associated to $E_{(\ell,m)}(K)$ are canonical K-algebra homomorphisms $\pi_\ell: E_{(\ell,m)}(K) \to M_\ell(K)$ and $\pi_m: E_{(\ell,m)}(K) \to M_m(K)$.

(ii) Viewing $K^{\ell+m}$ as left $E_{(\ell,m)}(K)$-module, identify K^ℓ with the submodule comprised of those column vectors having only zero entries below the ℓth position. Further identify K^m with the $E_{(\ell,m)}(k)$-module factor $K^{\ell+m}/K^\ell$.

(iii) Set

$$T_{\ell,m}(K) = \begin{bmatrix} 0 & M_{\ell\times m}(K) \\ 0 & 0 \end{bmatrix},$$

the Jacobson radical of $E_{(\ell,m)}(K)$.

3.4. For the remainder of this section, assume that $\rho : \; A \to E_{(\ell,m)}(K)$ is a representation, that $\Lambda = K\rho(A)$, and that J is the Jacobson radical of Λ. Also, let τ be an inner K-algebra automorphism of $M_{\ell+m}(K)$ such that $\tau(E_{(\ell,m)}(K)) \subseteq E_{(\ell,m)}(K)$. Of course, $\tau\rho$ will be a representation of A equivalent to ρ.

3.5. (i) If the compositions $\pi_{\ell\rho}$ and $\pi_{m\rho}$ are both irreducible, we will say that ρ is an (ℓ, m)*-extension of irreducible representations*; we will further say that ρ is a *self extension* when $\pi_{m\rho}$ and $\pi_{\ell\rho}$ are equivalent representations (and so $\ell = m$).

(ii) An (ℓ, m)-extension of irreducible representations *splits* if it is semisimple. It easily follows from standard results that every at-most n-dimensional representation of A is semisimple if and only if all (ℓ, m)-extensions of irreducible representations of A split, for all choices of $\ell + m \leqslant n$.

3.6 Lemma. *Assume that ρ is a nonsplit (ℓ, m)-extension of irreducible representations.*

(i) $J = T_{(\ell,m)}(K)$.

(ii) *Suppose that ρ is a self extension. Then we can choose τ such that*

$$\tau(\Lambda) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in M_\ell(K), \; b \in M_{\ell\times m}(K) \right\}.$$

(iii) *Suppose that ρ is not a self extension. Then $\Lambda = E_{(\ell,m)}(K)$.*

Proof. By considering the composition series $0 \subsetneq K^\ell \subsetneq K^{\ell+m}$, we see that $J \subseteq T_{(\ell,m)}(K)$.

Being a nonzero $M_\ell(K) - M_m(K)$-bimodule, J is a nonzero left module over

$$M_\ell(K) \otimes_K (M_m(K))^{\mathrm{op}} \cong M_{\ell m}(K).$$

Consequently, $\dim_K J \geqslant \ell m$, and so $J = T_{(\ell,m)}(K)$. Part (i) follows. Parts (ii) and (iii) follow easily from (i). $\square$

3.7 Lemma. (i) *Suppose that ρ is semisimple. Then $SupDiag_{\ell+m} \notin \tau(\Lambda)$.*

(ii) *Suppose that ρ is an (ℓ, m)-extension of irreducible representations. Then ρ does not split if and only if $SupDiag_{\ell+m} \in \tau(\Lambda)$ for some choice of τ.*

Proof. (i) The semisimplicity of ρ implies that Λ embeds into $M_\ell(K) \oplus M_m(K)$. Therefore, the maximum index of nilpotence of elements in Λ is less than $\ell + m$.

(ii) The "only if" statement follows from (3.6), and the "if" statement follows from (i). $\square$

3.8. The following notation will be used in the procedures presented in (3.9), (3.10), and (4.2).

(i) For positive integers ℓ, m, r, we will be $\mathbf{b}_r(\ell, m)$ denote the $(\ell+m) \times (\ell+m)$ matrix whose

$$ij\text{th entry} = \begin{cases} \text{the indeterminate } x_{ij}(r) & \text{if } i \leqslant \ell \text{ or } j \geqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For positive integers ℓ, m, s, we will let $\mathbf{U}(\ell, m, s)$ denote the set of all products $\mathbf{a}_1^{i_i} \cdots \mathbf{a}_p^{i_p}$ such that

$$\mathbf{a}_1, \ldots, \mathbf{a}_p \in \{I_{\ell+m}, \mathbf{b}_1(\ell, m), \ldots, \mathbf{b}_s(\ell, m)\}, \; i_1 + \cdots + i_p \leqslant \mathscr{P}(\ell + m),$$
$$\text{and} \quad i_1, \ldots, i_p < \ell + m.$$

Furthermore, temporarily letting $\mathbf{U} = \mathbf{U}(\ell, m, s)$, we will let $\pi_\ell(\mathbf{U})$ denote $\{\pi_\ell(\mathbf{u}) : \mathbf{u} \in \mathbf{U}\}$ and $\pi_m(\mathbf{U})$ denote $\{\pi_m(\mathbf{u}) : \mathbf{u} \in \mathbf{U}\}$.

3.9. *Semisimplicity test*: (Assume that k is computable and that K is the algebraic closure of k.) We now describe a test for deciding whether every at-most n-dimensional representation of R is semisimple. That the procedure works as stated follows directly from (3.2). Retain the notation of (3.8), and let $x_1, x_2, \ldots$ be indeterminates.

Input: $f_1, \ldots, f_t \in k\{X_1, \ldots, X_s\}$, positive integer n
Output: "all semisimple" if every at-most-n-dimensional representation
of $k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$ is semisimple; "not all semisimple" otherwise
Begin
For $1 \leqslant \ell < m \leqslant n$ *do*:
$q := \ell^2 + \ell m + m^2$
$\mathbf{V}$:= set of subsets of $\mathbf{U}(\ell, m, s)$ having cardinality q
$W := 0$
While $\mathbf{V} \neq \emptyset$ and $W = 0$ *do*:
Choose $\mathbf{V}_i = \{\mathbf{u}_1, \ldots, \mathbf{u}_q\} \in \mathbf{V}$
If the entries of
$$x_1\mathbf{u}_1 + \cdots + \mathbf{x}_q\mathbf{u}_q - SupDiag_{\ell+m}$$
$$f_1(\mathbf{b}_1(\ell, m), \ldots, \mathbf{b}_s(\ell, m)), \ldots, f_t(\mathbf{b}_1(\ell, m), \ldots, \mathbf{b}_s(\ell, m)),$$
have a common zero over K then $W := 1$
Else $\mathbf{V} := \mathbf{V} \backslash \{\mathbf{V}_i\}$
End
End
If $W = 0$ *then return* "all semisimple"
Else return "not all semisimple"
End

Note that the subtest within the while loop involves $(t + 1)_q$ polynomials in $(s + 1)_q - (\ell^2 - \ell)/2 - (m^2 - m)/2$ variables. The degrees of q of these polynomials will be bounded by $\mathscr{P}(\ell + m) + 1$, and the remaining degrees will be bounded by d. Following (2.6), $\mathscr{D} \leqslant d^{uq}(\mathscr{P}(\ell + m) + 1)^q$.

3.10. *Nonsplit* (ℓ, m)-*extension test*: (Assume that k is computable and that K is the algebraic closure of k.) We now combine (3.7) with (2.5) to devise a procedure for deciding, for fixed ℓ and m, whether R has a nonsplit (ℓ, m)-extension of irreducible representations. Retain the notation of (3.8), and let v, w, and $y_1, y_2, \ldots$ be indeterminates. (Note: While the following algorithm works as stated, it would be reasonable in general to first check for existence of ℓ-dimensional and m-dimensional irreducible representations, following (2.5).)

Input: $f_1, \dots, f_t \in k\{X_1, \dots, X_s\}$, positive integers ℓ and m
Output: "yes" if there exists a nonsplit (ℓ, m)-extension of irreducible representations of $k\{X_1, \dots, X_s\}/\langle f_1, \dots, f_t\rangle$; "no" otherwise
Begin
$q := \ell^2 + \ell m + m^2$
$\mathbf{U} := \mathbf{U}(\ell, m, s)$
$\mathbf{V}$:= set of subsets of $\pi_\ell(\mathbf{U})$ having cardinality $2(\ell - 1)$
$\mathbf{W}$:= set of subsets of $\pi_m(\mathbf{U})$ having cardinality $2(m - 1)$
$\mathbf{Y}$:= set of subsets of $\mathbf{U}$ having cardinality q
$\mathbf{T} := \mathbf{U} \times \mathbf{V} \times \mathbf{U} \times \mathbf{W} \times \mathbf{Y}$
$Z := 0$
While $Z = 0$ and $\mathbf{T} \neq \emptyset$ *do*:
Choose $\mathbf{T}_i = (\mathbf{v}_0, \{\mathbf{v}_1, \dots, \mathbf{v}_{2(\ell-1)}\}, \mathbf{w}_0, \{\mathbf{w}_1, \dots, \mathbf{w}_{2(m-1)}\}, \{\mathbf{y}_1, \dots, \mathbf{y}_q\}) \in \mathbf{T}$
If the entries of

$$v\, trace(\mathbf{v}_0 s_{2(\ell-1)}(\mathbf{v}_1, \dots, \mathbf{v}_{2(\ell-1)})) - 1,$$
$$w\, trace(\mathbf{w}_0 s_{2(m-1)}(\mathbf{w}_1, \dots, \mathbf{w}_{2(m-1)})) - 1,$$
$$y_1 \mathbf{y}_1 + \cdots + y_q \mathbf{y}_q - SupDiag_{\ell+m}$$
$$f_1(\mathbf{b}_1(\ell, m), \dots, \mathbf{b}_s(\ell, m)), \dots, f_t(\mathbf{b}_1(\ell, m), \dots, \mathbf{b}_s(\ell, m)),$$

have a common zero **then** $Z := 1$
Else
$\mathbf{T} := \mathbf{T} \backslash \{\mathbf{T}_i\}$
End
If $Z = 1$ then *return* "yes"
Else return "no"
End

The subtest within the while loop involves $(t + 1)q + 2$ polynomials in $(s + 1)q + 2 - (\ell^2 - \ell)/2 - (m^2 - m)/2$ variables. The degrees of q of the polynomials are bounded by $\mathscr{P}(\ell + m) + 1$, the degree of one of the polynomials is bounded by $\mathscr{P}(\ell)^{2\ell-1} + 1$, and the degree of one of the polynomials is bounded by $\mathscr{P}(m)^{2m-1} + 1$. The remaining degrees are bounded by d. Following (2.6), $\mathscr{D} \leqslant (\mathscr{P}(\ell)^{2\ell-1} + 1)(\mathscr{P}(m)^{2m-1} + 1)(\mathscr{P}(\ell + m) + 1)^q d^{uq}$.

4. Nonsplit extensions of distinct irreducible representations

4.1 Proposition. *Let A be a k-algebra. The following statements are equivalent*: (i) *There exists a nonsplit (ℓ, m)-extension of inequivalent irreducible representations of A for some $\ell + m \leqslant n$.* (ii) *For some $\ell + m \leqslant n$, there exists a representation $\rho : A \to E_{(\ell,m)}(K)$ for which $SupDiag_{\ell+m}, I_\ell \in K\rho(A)$.*

Proof. (i)⇒(ii): Follows from (3.6iii).

(ii)⇒(i): Set $\Lambda = K_\rho(A)$. If $K^{\ell+m}$ is decomposable as a left Λ-module, then Λ embeds into $M_\mu(K) \oplus M_\nu(K)$, for some $\mu, \nu < \ell + m$, implying that Λ cannot contain an element whose index of nilpotence is $\ell+m$. Therefore, since $SupDiag_{\ell+m} \in \Lambda$, we see that $K^{\ell+m}$ is an indecomposable Λ-module.

Now let M be the Λ-submodule $\Lambda I_\ell K^{\ell+m}$ of $K^{\ell+m}$, and set $N = K^{\ell+m}/M$. Since Λ is a subalgebra of $E_{(\ell,m)}(K)$, we see that both M and N are nonzero. It follows from the preceding paragraph that the exact sequence $0 \to M \to K^{\ell+m} \to N \to 0$ is a nonsplit extension of Λ-modules. Therefore, there exists a nonsplit extension of L' by L for some simple Λ-module subfactor L of M and simple Λ-module subfactor L' of N. Note, however, that I_ℓ acts as the identity on L and that $I_\ell L' = 0$. Therefore, L and L' cannot be isomorphic as Λ-modules.

Consequently, for some $1 \leqslant \ell' \leqslant \ell$ and $1 \leqslant m' \leqslant m$, there exists a nonsplit (ℓ', m')-extension of inequivalent irreducible representations $\rho' : A \to E_{(\ell',m')}(K)$. $\square$

4.2 *Nonsplit non-self extension test*: (Assume that k is computable and that K is the algebraic closure of k.) Retain the notation of (3.8), and let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be indeterminates. We can now describe a test, as follows, for determining the existence of nonsplit extensions of inequivalent irreducible representations. That the procedure works as stated follows directly from (4.1).

Input: $f_1, \ldots, f_t \in k\{X_1, \ldots, X_s\}$, positive integer n
Output: "yes" if there exists a nonsplit (ℓ, m)-extension of inequivalent irreducible representations for some $\ell + m \leqslant n$; "no" otherwise
Begin
  For $1 \leqslant \ell < m \leqslant n$ *do*:
  $q := \ell^2 + \ell m + m^2$
  $\mathbf{V}$:= set of subsets of $\mathbf{U}(\ell, m, s)$ having cardinality q
  $W := 0$
    While $\mathbf{V} \neq \emptyset$ and $W = 0$ *do*:
    Choose $\mathbf{V}_i = \{\mathbf{v}_1, \ldots, \mathbf{v}_q\} \in \mathbf{V}$
    If the entries of
    $x_1\mathbf{v}_1 + \cdots + x_q\mathbf{v}_q - SupDiag_{\ell+m}$
    $y_1\mathbf{v}_1 + \cdots + y_q\mathbf{v}_q - I_\ell$
    $f_1(\mathbf{b}_1(\ell, m), \ldots, \mathbf{b}_s(\ell, m)), \ldots, f_t(\mathbf{b}_1(\ell, m), \ldots, \mathbf{b}_s(\ell, m))$,
    have a common zero over K then $W := 1$
    Else$\mathbf{V}$: $\mathbf{V} \backslash \{\mathbf{V}_i\}$
    End
  End
If $W = 1$ *then return* "yes"
Else return "no"
End

The subtest within the while loop involves $(t+2)q$ polynomials in $(s+2)q - (\ell^2 - \ell)/2 - (m^2 - m)/2$ variables. The degrees of $2q$ of these polynomials will be bounded by $\mathscr{P}(\ell + m) + 1$, and the remaining degrees will be bounded by d. Following (2.6), $\mathscr{D} \leqslant d^{uq}(\mathscr{P}(\ell + m) + 1)^{2q}$.

4.3. We leave to the reader the construction of a test that decides the existence of a nonsplit (ℓ, m)-extension of inequivalent irreducible representations, for fixed ℓ and m.

5. Example: nonsplit extensions of one-dimensional representations

As an elementary (and easy) illustration of the methods of the preceding sections, we consider the case when $\ell = m = 1$. Nonsplit extensions of one-dimensional representations play an important role in the study of many natural classes of finitely presented algebras—for example, in the study of solvable Lie algebras (cf., e.g., [6]) and quantum function algebras (e.g., [7]).

Assume that k is computable and that K is the algebraic closure of k. Recall that $R = k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$.

5.1. (i) For $1 \leqslant r \leqslant s$, set

$$\mathbf{b}_r = \begin{bmatrix} x_{11}(r) & x_{12}(r) \\ \mathbf{0} & x_{22}(r) \end{bmatrix}.$$

(ii) Following (2.5), and noting that $\mathscr{P}(2) \leqslant 3$, we set

$$\mathbf{V} = \{I_n\} \cup \{\mathbf{b}_1, \ldots, \mathbf{b}_s\} \cup \{\mathbf{b}_\alpha \mathbf{b}_\beta : \alpha \neq \beta\} \cup \{\mathbf{b}_\alpha \mathbf{b}_\beta \mathbf{b}_\gamma : \alpha \neq \beta \neq \gamma\}.$$

(iii) Let a_1, a_2, and a_3 be indeterminates. By (3.9), there exists a nonsplit extension of one-dimensional representations of R if and only if the polynomial entries of

$$f_1(\mathbf{b}_1, \ldots, \mathbf{b}_s), \ldots, f_t(\mathbf{b}_1, \ldots, \mathbf{b}_s), \quad a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have a common zero for some choice of distinct $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{V}$.

(iv) Let a_1, a_2, a_3, b_1, b_2, and b_3 be indeterminates. By (4.2), there exists a nonsplit extension of inequivalent one-dimensional representations of R if and only if the polynomial entries of

$$\begin{aligned} &f_1(b_1, \ldots, b_s), \ldots, f_t(b_1, \ldots, b_s), \\ &a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ &b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

have a common zero for some choice of distinct $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{V}$.

(v) If $s = 3$ then $|\mathbf{V}| = 22$, and there are $\binom{22}{3} = 1540$ cases to check in (iii) and (iv). It is not unusual for the first interesting cases of a given class of algebras to require three generators or fewer—well-known occurrences of this phenomenon include the enveloping algebra of sl_2, the enveloping algebra of the Heisenberg Lie algebra, and the three-dimensional regular algebras of [2,3].

5.2. We conclude by considering two concrete examples. All of the computations mentioned below were performed using Macaulay2 on a personal computer (4 GB RAM). Let

$$\mathbf{x} = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_{11} & z_{12} \\ 0 & z_{22} \end{bmatrix}.$$

(i) Set

$$R = \mathbb{Q}\{X, Y, Z\}/\langle XY - YX - Z, XZ - ZX, YZ - ZY\rangle,$$

the universal enveloping algebra of the (nilpotent) Heisenberg Lie algebra. It follows from well known abstract arguments that R does not have nonsplit extensions of inequivalent one-dimensional representations but does have nonsplit self extensions of one-dimensional representations; see, for example, [10].

Evaluating all 1540 cases, we were easily able to check that the entries of

$$\begin{aligned}
&\mathbf{xy} - \mathbf{yx} - \mathbf{z}, \quad \mathbf{xz} - \mathbf{zx}, \quad \mathbf{yz} - \mathbf{zy} \\
&a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
&b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

have no common zeros for indeterminates $a_1, a_2, a_3, b_1, b_2, b_3$ and all choices of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{V}$. We thus recovered the fact that R has no nonsplit extensions of distinct one-dimensional representations. Next, evaluating all 1540 cases of

$$\mathbf{xy} - \mathbf{yx} - \mathbf{z}, \quad \mathbf{xz} - \mathbf{zx}, \quad \mathbf{yz} - \mathbf{zy} \quad a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we found that there exists a common zero—indicating the presence of a nonsplit self extension—in 980 instances.

(ii) Now set

$$R = \mathbb{Q}\{X, Y, Z\}/\langle XY - YX - Y, XZ - ZX, YZ - ZY\rangle,$$

an enveloping algebra of a solvable-but-not-nilpotent Lie algebra. Here it follows from abstract considerations that R has nonsplit self and non-self extensions of one-dimensional representations (again see e.g., [10]).

Testing all 1540 cases of

$$\mathbf{xy} - \mathbf{yx} - \mathbf{y}, \quad \mathbf{xz} - \mathbf{zx}, \quad \mathbf{yz} - \mathbf{zy}, \quad a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{V}$, we found that there exists a common zero—indicating the presence of a nonsplit extension—in 1539 instances. Only in the case $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{I_2, \mathbf{yxy}, \mathbf{yzy}\}$

did there not exist a common zero. Testing

$$\begin{aligned}
&\mathbf{xy} - \mathbf{yx} - \mathbf{y}, \quad \mathbf{xz} - \mathbf{zx}, \quad \mathbf{yz} - \mathbf{zy} \\
&a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
&b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

we found that there exists a common zero—indicating the presence of a nonsplit non-self extension—in 650 instances.

5.3. Unfortunately, at this time, we are unaware of general methods for significantly simplifying the computations involved in the procedures described in this paper. Systematic studies of more practical approaches to these or related tests are left for future work.

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